Homogenization of stable-like operators

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Outline

1. **Aim**
2. **Symmetric setting: ergodic medium**
   - Framework: Dirichlet form
   - Main results
3. **Non-symmetric case: periodic coefficient**
   - Framework: operator
   - Main result

Homogenization

(1)

$$L = \sum_{1 \leqslant i,j \leqslant d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

(2) Oscillating coefficients

$$L^\varepsilon = \sum_{1 \leqslant i,j \leqslant d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right), \quad \varepsilon > 0.$$

(3) Homogenization

$$L^\varepsilon \to \bar{L}, \quad \varepsilon \to 0,$$

where $\bar{L}$ is with constant coefficient.

- (i) Periodic homogenization: $a_{ij}(x)$ is a periodic function (defined on $\mathbb{T}^d$).
- (ii) Stochastic homogenization (in a stationary, ergodic random media): $a_{ij}(x;\omega) = a_{ij}(\tau_x \omega)$, where $\{\tau_x\}_{x\in\mathbb{R}^d}$ is a measurable group of transformations defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\{\tau_x\}_{x\in\mathbb{R}^d}$ is stationary and ergodic.

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$$\mathcal{E}^{\varepsilon}(f,g) = \sum_{1\leqslant i,j\leqslant d} \int a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial f(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_j}\, dx.$$

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Known results for diffusive homogenization

- periodic environment [Bensoussan, Lions and Papanicolaou 1975], [Tatar 1976]
- ergodic environment [Kozlov 1979], [Papanicolaou and Varadhan 1979]

$$L^{\varepsilon,\omega} \to \bar{L} = \sum_{1\leqslant i,j\leqslant d} \bar{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad \varepsilon \to 0, a.s. \omega \in \Omega$$

where $\bar{a}_{ij} = \mathbb{E}[\sum_{1\leqslant k\leqslant d} a_{ij}(0)(\delta_{kj} + \psi_j^k(0))]$, $\psi_j^k(x;\omega) := \frac{\partial}{\partial x_k}\chi_j(x;\omega)$,

$$L\chi_j(x;\omega) = -\sum_{1\leqslant i\leqslant d} \frac{\partial}{\partial x_i} a_{ij}(x;\omega).$$

- Existence of corrector: L^2 integrability of coefficients.

Question: Homogenization problem for stable-like operators

(1) What kind of stable-like operator L we will consider?

(2) How can we do the homogenization? What kind of scaling we will choose?

(3) What expression of the limiting operator $\bar{L}$?

What kind of stable-like operator L under scaling with the limiting operator $\bar{L}$?

Let $(X_t)_{t\geqslant 0}$ be a α-stable-like process (not only α-stable Lévy process and, in general, not having the scaling property) with generator as follow

- Symmetric setting:

$$Lf(x) = p.v. \int (f(y) - f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}}\, dy$$

where $c(x,y) = c(y,x)$ for all $x, y \in \mathbb{R}^d$.

- Non-symmetric setting:

$$Lf(x) = p.v. \int (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz.$$

What kind of homogenization: For any $\varepsilon > 0$ and $t > 0$, let $X_t^{(\varepsilon)} := \varepsilon X_{\varepsilon^{-\alpha} t}$.
Question: We will consider that, under some assumptions, $(X_t^{(\varepsilon)})_{t\geqslant 0}$ converges to some $(\bar{X}_t)_{t\geqslant 0}$ as $\varepsilon \to 0$ and what is the expression for its infinitesimal generator.

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Known results

- M. Tomisaki: Homogenization of cádlág processes, *J. Math. Soc. Japan*, **44** (1992), 281–305.
- M. Kassmann, A. Piatnitski and E. Zhizhina: Homogenization of Lévy-type operators with oscillating coefficients, to appear in *SIAM J. Math. Anal.*
- R.L. Schilling and T. Uemura: Homogenization of symmetric Lévy processes on $\mathbb{R}^d$, arXiv:1808.01667
- R.W. Schwab: Stochastic homogenization for some nonlinear integro-differential equations, *Comm. Partial Differential Equations*, **38** (2013), 171–198.
- R.W. Schwab: Periodic homogenization for nonlinear integro-differential equations, *SIAM J. Math. Anal.*, **42** (2010) 2652–2680.

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- Z.Q. Chen, P. Kim and T. Kumagai: Discrete approximation of symmetric jump processes on metric measure spaces, *Proba. Theory Relat. Fields*, 155, 2013, 703–749.
- X. Chen, T.Kumagai and J. Wang: Random conductance models with stable-like jumps I: Quenched invariance principle, arXiv:1805.04344.
- X. Chen, T. Kumagai and J. Wang: Random conductance models with stable-like jumps: heat kernel estimates and Harnack inequalities, arXiv:1808.02178.
- J.Q. Duan, Q. Huang and R.M. Song: Homogenization of stable-like Feller processes, arXiv:1812.11624.

Outline

Ergodic Environment

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a group of transformation $\{\tau_x\}_{x\in\mathbb{R}^d}$ such that

- $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$ and $x \in \mathbb{R}^d$; (Stationary)
- If $A \in \mathcal{F}$ and $\tau_x A = A$ for all $x \in \mathbb{R}^d$, then $\mathbb{P}(A) \in \{0, 1\}$; (Ergodic)
- The function $(x, \omega) \mapsto \tau_x\omega$ is measurable; (Measurable)

Symmetric stable-like operator L in random medium

Let $(X_t^\omega)_{t\geqslant 0}$ be a symmetric α-stable-like process with generator as follow

- $$L^\omega f(x) = p.v. \int (f(y) - f(x)) \frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\, dy$$

  where $c(x,y;\omega) = c(y,x;\omega)$ for all $x, y \in \mathbb{R}^d$.

- Non-local Dirichlet form:

  $$\begin{aligned}\mathcal{E}^\omega(f,g) &= -\int f(x) L^\omega g(x)\, dx \\ &= \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) \frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\, dx\, dy\end{aligned}$$

  on $L^2(\mathbb{R}^d; dx)$.

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Non-local symmetric Dirichlet form: starting point

- A little more general, allowing the degenerate reference measure:

$$\mathcal{E}^\omega(f,g) := \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f(x)-f(y))(g(x)-g(y))\frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\,dx\,dy$$

on $L^2(\mathbb{R}^d;\mu(x;\omega)\,dx)$.

- The corresponding operator on $L^2(\mathbb{R}^d;\mu(x;\omega)\,dx)$:

$$L^\omega f(x) = \frac{1}{\mu(x;\omega)}\int_{\mathbb{R}^d}(f(y)-f(x))\,\frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\,dy.$$

- Translation invariance of coefficients: $c(x+z,y+z;\omega)=c(x,y;\tau_z\omega)$, $\mu(x+z;\omega)=\mu(x;\tau_z\omega)$.

Scaling processes

For any $\varepsilon > 0$, set $X^{\varepsilon,\omega} = (X_t^{\varepsilon,\omega})_{t\geqslant 0} := (\varepsilon X^{\omega}_{\varepsilon^{-\alpha}t})_{t\geqslant 0}$.

Lemma

The process $X^{\varepsilon,\omega}$ enjoys a symmetric measure $\mu^{\varepsilon,\omega}(dx) = \mu\left(\frac{x}{\varepsilon};\omega\right)dx$, and the associated regular Dirichlet form $(\mathcal{E}^{\varepsilon,\omega}, \mathcal{F}^{\varepsilon,\omega})$ on $L^2(\mathbb{R}^d;\mu^{\varepsilon,\omega}(dx))$ is given by

$$\mathcal{E}^{\varepsilon,\omega}(f,g) = \frac{1}{2}\iint (f(x)-f(y))(g(x)-g(y))\frac{c\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon};\omega\right)}{|x-y|^{d+\alpha}}\,dx\,dy.$$

Limiting Dirichlet form:

$$\bar{\mathcal{E}}(f,g) = \frac{1}{2}\iint (f(y)-f(x))(g(y)-g(x))\frac{\bar{k}(x-y)}{|x-y|^{d+\alpha}}\,dx\,dy,$$

where $\bar{k}(z) = \bar{k}(-z)$ for all $z \in \mathbb{R}^d$.

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Assumption

- Assumption $(A\text{-}\mu)$ Suppose $\mathbb{E}\,\mu(0;\omega)=1$.

$$\int f(x)\mu(x/\varepsilon;\omega)\,dx = \int f(x)\mu(0;\tau_{x/\varepsilon}\omega)\,dx$$
$$\to \int f(x)\mathbb{E}\mu(0;\omega)\,dx = \int f(x)\,dx.)$$

- What we need is

$$c\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon};\omega\right) \dashrightarrow \bar{k}(x-y), \quad \varepsilon\to 0.$$

- **Difficulty**

$$c\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon};\omega\right) = c\left(0,\frac{y-x}{\varepsilon};\tau_{\frac{x}{\varepsilon}}\omega\right) \dashrightarrow \bar{k}(x-y)???, \quad \varepsilon\to 0.$$

$\mathbb{E}c(0,z/\varepsilon;\omega)?$ *or* $c(0,z/\varepsilon;\omega)?$

- **Difficulty** The jumping kernel is not L^2 integrable, not easy to construct corrector.

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- [Z.Q. Chen, P. Kim and T.Kumagai 2013] Homogenization for random conductance model on $\mathbb{Z}^d$ with mutually independent conductance.
- [M. Kassmann, A. Piatnitski and E. Zhizhina 2018]
  If $c(x,y;\omega) = \sigma_1(x;\omega)\sigma_1(y;\omega) = \sigma_1(0;\tau_x\omega)\sigma_1(0;\tau_y\omega)$, $0 < K_1 \leqslant c(x,y;\omega) \leqslant K_2$ and $\mu(x;\omega) = \frac{\sigma_2(0;\tau_x\omega)}{\sigma_1(0;\tau_x\omega)}$, then for a.s. $\omega \in \Omega$ and $f \in C_c^2(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} |U_\lambda^{\varepsilon,\omega} f(x) - \bar{U}_\lambda f(x)|^2 dx = 0,$$

where $\bar{U}_\lambda f$ is the resolvent associated with

$$\overline{\mathcal{E}}(f,g) = \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) \frac{(\mathbb{E}[\sigma_1(0;\omega)])^2}{|x-y|^{d+\alpha}}\, dx\, dy$$

Known results

- [X. Chen, T.Kumagai and J. Wang 2018] Quenched invariance principle (limit of process with initial point fixed), large scale parabolic regularity for symmetric stable-like process on random conductance model on $\mathbb{Z}^d$ with mutually independent conductance.
- [X. Chen, T.Kumagai and J. Wang 2018] Large time heat kernel estimates

$$C_1\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big) \leqslant p^\omega(t,x,y) \leqslant C_2\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big),$$

$$\forall\, t > (R_x(\omega) \vee |x-y|)^\theta$$

Question:

- What is the case for the $c(x, y; \omega)$ with more general form? Under this case, what is the expression for $\overline{\mathcal{E}}$?
- Could we prove the result without uniform ellipticity condition $0 < K_1 \leqslant c(x, y; \omega) \leqslant K_2 < \infty$?

Outline

Assumptions on the coefficient $c(x, y; \omega)$

(Form-1) : There exists a measurable function $k : \mathbb{R}^d \times \Omega \to [0, \infty)$ such that

- $c(x, y; \omega) = k(y - x; \tau_x \omega) + k(x - y; \tau_y \omega)$,
-
$$\sup_{z \in \mathbb{R}^d} \mathbb{E} k(z; \cdot)^2 < \infty,$$
- There are constants $l > d$ and $C_0 > 0$ so that for any z_1, z_2 and $x \in \mathbb{R}^d$,
$$\Big| \mathbb{E}\big(k(z_1; \cdot) k(z_2; \tau_x \cdot)\big) - \mathbb{E} k(z_1; \cdot) \cdot \mathbb{E} k(z_2; \cdot) \Big| \leqslant C_0 \|k(z_1; \cdot)\|_{L^2(\Omega;\mathbb{P})} \|k(z_2; \cdot)\|_{L^2(\Omega;\mathbb{P})} \big(1 \wedge |x|^{-l}\big),$$
-
$$\lim_{\varepsilon \to 0} \sup_{|z| \leqslant r} \left| \mathbb{E}\, c\left(0, \frac{z}{\varepsilon}; \omega\right) - K(z) \right| = 0, \quad r > 0,$$

  for a function $K(z)$ satisfying $0 < K_1 \leqslant K(z) \leqslant K_2 < \infty$.

Assumptions on the coefficient $c(x, y; \omega)$

(Form-2) : There exists a measurable function $k : \mathbb{R}^d \times \Omega \to [0, \infty)$ such that $c(x, y; \omega) = k(y - x; \tau_x\omega) + k(x - y; \tau_y\omega)$, and for a.s. $\omega \in \Omega$ and $r > 0$,

$$\lim_{\varepsilon \to 0} \sup_{|x| \leqslant r, |z| \leqslant r} \left| k\left(\frac{x}{\varepsilon}; \tau_{\frac{z}{\varepsilon}}\omega\right) - \bar{k}(x; \tau_{\frac{z}{\varepsilon}}\omega) \right| = 0,$$

where $\bar{k}$ is a measurable function such that the function $0 < C_1 \leqslant \mathbb{E}\,\bar{k}(z; \cdot) \leqslant C_2 < \infty$ for some positive constants C_1, C_2.

- **(Form-2)** includes the scaling invariant condition $k(z; \omega) = k\left(\frac{z}{\varepsilon}; \omega\right)$ adopted in [Schwab 2014].
- Another model: $k(z; \omega)$ is periodic with respect to z.

Assumptions on the coefficient $c(x, y; \omega)$

(Bound-1) There are nonnegative random variables $\Lambda_1(\omega) \leqslant \Lambda_2(\omega)$ such that for a.s. $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$,

$$\Lambda_1(\tau_x\omega) + \Lambda_1(\tau_y\omega) \leqslant c(x, y; \omega) \leqslant \Lambda_2(\tau_x\omega) + \Lambda_2(\tau_y\omega),$$

and for some $p > 1$, $q > 1$,

$$\mathbb{E}\Big(\Lambda_1^{-q}(\cdot) + \Lambda_2^p(\cdot) + \mu^p(0; \cdot)\Big) < \infty.$$

Assumptions on the coefficient $c(x, y; \omega)$

(Bound-2) There are nonnegative random variables $\Lambda_1(\omega) \leqslant \Lambda_2(\omega)$ such that for a.s. $\omega \in \Omega$,

$$\Lambda_1(\tau_x\omega)\Lambda_1(\tau_y\omega) \leqslant c(x, y; \omega) \leqslant \Lambda_2(\tau_x\omega)\Lambda_2(\tau_y\omega), \quad x, y \in \mathbb{R}^d,$$

and

$$\mathbb{E}\Big(\Lambda_1^{-q}(\cdot) + \Lambda_2^{p}(\cdot) + \mu^{p/2}(0; \cdot)\Big) < \infty$$

for some constants $p > 2$ and $q > 2$.

Main Theorem

Theorem

Suppose that **(Form-1)** *and* **(Bound-1)** *hold. Then, for a.s.* $\omega \in \Omega$, *any* $f \in C_c^\infty(\mathbb{R}^d)$ *and* $\lambda > 0$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |U_\lambda^{\varepsilon,\omega} f(x) - \bar{U}_\lambda f(x)|^2 \mu\left(\frac{x}{\varepsilon}; \omega\right) dx = 0,$$

where $\bar{U}_\lambda f$ *is the resolvent associated with*

$$\overline{\mathcal{E}}(f,g) = \frac{1}{2} \iint (f(x) - f(y))(g(x) - g(y)) \frac{K(x-y) + K(y-x)}{|x-y|^{d+\alpha}} \, dx \, dy.$$

Mosco Convergence

Mosco convergence with changing measures ([Kuwae-Shioya 2003], [Kolesnikov 2005]):

- For every sequence $\{f_n\}_{n\geqslant 1}$ on $L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ converging weakly to $f\in L^2(\mathbb{R}^d;dx)$,

$$\liminf_{n\to\infty}\mathcal{E}^{\varepsilon_n,\omega}(f_n,f_n)\geqslant\overline{\mathcal{E}}(f,f).$$

- For any $f\in L^2(\mathbb{R}^d;dx)$, there is $\{f_n\}_{\geqslant 1}\subset L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ converging strongly to f such that

$$\limsup_{n\to\infty}\mathcal{E}^{\varepsilon_n,\omega}(f_n,f_n)\leqslant\overline{\mathcal{E}}(f,f).$$

Main Theorem

Theorem

Suppose that **(Form-2)** *and* **(Bound-1)** *hold. Then, for a.s.* $\omega \in \Omega$, *any* $f \in C_c^\infty(\mathbb{R}^d)$ *and* $\lambda > 0$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |U_\lambda^{\varepsilon,\omega} f(x) - \bar{U}_\lambda f(x)|^2 \mu\Big(\frac{x}{\varepsilon}; \omega\Big) dx = 0,$$

where $\bar{U}_\lambda f$ *is the resolvent associated with*

$$\overline{\mathcal{E}}(f, g) = \frac{1}{2} \iint (f(x) - f(y))(g(x) - g(y)) \frac{\mathbb{E}[\bar{k}(x-y) + \bar{k}(y-x)]}{|x-y|^{d+\alpha}} \, dx \, dy.$$

Main Theorem

Theorem

Suppose that $c(x,y;\omega)=\sigma_1(0;\tau_x\omega)\sigma_1(0;\tau_y\omega)$, $\mu(x;\omega)=\frac{\sigma_2(0;\tau_x\omega)}{\sigma_1(0;\tau_x\omega)}$ *and* **(Bound-2)** *hold. Then, for a.s.* $\omega\in\Omega$, *any* $f\in C_c^\infty(\mathbb{R}^d)$ *and* $\lambda>0$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}|U_\lambda^{\varepsilon,\omega}f(x)-\bar{U}_\lambda f(x)|^2\mu\Big(\frac{x}{\varepsilon};\omega\Big)dx=0,$$

where $\bar{U}_\lambda f$ *is the resolvent associated with*

$$\overline{\mathcal{E}}(f,g)=\frac{1}{2}\iint(f(x)-f(y))(g(x)-g(y))\frac{\mathbb{E}[\sigma_1(\omega)]^2}{|x-y|^{d+\alpha}}\,dx\,dy.$$

Outline

Non-symmetric setting

- Let $\alpha \in (0,1)$. Consider the following operator acting on $C_b^2(\mathbb{R}^d)$:

$$
\begin{aligned}
Lf(x) &= \int_{\mathbb{R}^d} (f(y)-f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}}\, dy \\
&= \int_{\mathbb{R}^d} (f(y)-f(x)) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz.
\end{aligned}
$$

  (Note that, $c(x,y)$ is not symmetric with respect to (x,y) and $k(x,z) = c(x,x+z)$.)

- Coefficients: Let $c(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ be periodic with respect to both variables such that

  (i) $0 < C_1 \leqslant c(x,y) \leqslant C_2 < \infty$ for all $x, y \in \mathbb{R}^d$.
  (ii) $k(\cdot,\cdot) \in C_b^{2,1}(\mathbb{R}^d \times \mathbb{R}^d)$.

- Let $(X_t)_{\geqslant 0}$ be the process associated with L.

Known Results

- [M. Kassmann, A. Piatnitski and E. Zhizhina 2018]
  If $\alpha \in (0,1)$, then $\varepsilon X_{\varepsilon^{-\alpha}\cdot} \to \bar{X}_{\cdot}$ with corresponding infinitesimal generator

$$\bar{L}f(x) = \int_{\mathbb{R}^d} \left(f(x+z) - f(x)\right) \frac{\bar{k}}{|z|^{d+\alpha}} dz,$$

  where $\bar{k} = \iint_{\mathbb{T}^d \times \mathbb{T}^d} k(y,z) dz \bar{\mu}(dy)$ with $\bar{\mu}$ being the invariant measure for $(X_t)_{t \geqslant 0}$.
- Question: What is the case for $\alpha \in [1,2)$.

Settings: periodic homogenization

- Let $\alpha \in (1,2)$. Consider the following operator acting on $C_b^2(\mathbb{R}^d)$:

$$Lf(x) = p.v. \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz$$
$$= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z\rangle) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz + \langle \nabla f(x), b_0(x)\rangle,$$

where

$$b_0(x) := \frac{1}{2} \int z \frac{(k(x,z) - k(x,-z))}{|z|^{d+\alpha}}\, dz, \quad x \in \mathbb{R}^d.$$

(Note that, here we do not require that $k(x,z) = k(x,-z)$ for all $x, z \in \mathbb{R}^d$.)

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- Let $\alpha \in (1,2)$. Consider the following operator acting on $C_b^2(\mathbb{R}^d)$:

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&= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z\rangle) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz + \langle \nabla f(x), b_0(x)\rangle,
\end{aligned}
$$

where

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b_0(x) := \frac{1}{2}\int z \frac{(k(x,z) - k(x,-z))}{|z|^{d+\alpha}}\, dz, \quad x \in \mathbb{R}^d.
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- Let $\alpha \in (1,2)$. Consider the following operator acting on $C_b^2(\mathbb{R}^d)$:

$$Lf(x) = p.v.\int_{\mathbb{R}^d} (f(x+z)-f(x))\frac{k(x,z)}{|z|^{d+\alpha}}\,dz$$
$$= \int_{\mathbb{R}^d} (f(x+z)-f(x)-\langle \nabla f(x), z\rangle)\frac{k(x,z)}{|z|^{d+\alpha}}\,dz + \langle \nabla f(x), b_0(x)\rangle,$$

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Non-symmetric α-stable-like processes

- Let $\alpha \in (1, 2)$.

$$Lf(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz$$
$$= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz + \langle \nabla f(x), b_0(x) \rangle.$$

We need the continuity of z to ensure the regularity of b_0.

- $$Lf(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz + \langle \nabla f(x), b(x) \rangle.$$

(Note that, for this the continuity of $k(x,z)$ with respect to z is not required. We only need to assume that b is bounded.)

- There exists a non-symmetric α-stable-like process $X := (X_t)_{t \geqslant 0}$, see Chen-Zhang (14',18').

- To establish the limit of the scaling process $(\varepsilon X_{\varepsilon^{-\alpha} t})_{t \geqslant 0}$.

Non-symmetric α-stable-like processes

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-

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  (Note that, for this the continuity of $k(x,z)$ with respect to z is not required. We only need to assume that b is bounded.)

- There exists a non-symmetric α-stable-like process $X := (X_t)_{t\geqslant 0}$, see Chen-Zhang (14',18').

- To establish the limit of the scaling process $(\varepsilon X_{\varepsilon^{-\alpha}t})_{t\geqslant 0}$.

Non-symmetric α-stable-like processes

- Let $\alpha = 1$.

$$
\begin{aligned}
Lf(x) &= \int_{\mathbb{R}^d} \left(f(x+z) - f(x)\right) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz \\
&= \int_{|z| \leqslant \frac{1}{\varepsilon}} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle\right) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz + \langle \nabla f(x), b_0^{\varepsilon}(x) \rangle,
\end{aligned}
$$

where $b_0^{\varepsilon}(x) = \frac{1}{2} \int_{|z| \leqslant \frac{1}{\varepsilon}} z \frac{k(x,z) - k(x,-z)}{|z|^{d+\alpha}} dz.$

Non-symmetric α-stable-like processes

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$$Lf(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz$$
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where $b_0^{\varepsilon}(x) = \frac{1}{2} \int_{|z| \leqslant \frac{1}{\varepsilon}} z \frac{k(x,z) - k(x,-z)}{|z|^{d+\alpha}} dz.$

Outline

Main result

Theorem

If $\alpha \in (1, 2)$, there exist a vector $\bar{b}_0 \in \mathbb{R}^d$ and a constant $\bar{k}_0 > 0$ such that the process $\{\varepsilon(X_{\varepsilon^{-\alpha}t} - \varepsilon^{-\alpha}\bar{b}_0 t)\}_{t \geqslant 0}$ converges, as $\varepsilon \to 0$, in the Skorokhod topology to a rotationally invariant α-stable Lévy process $\bar{X}$ with the generator

$$\bar{L}f(x) = \int (f(x+z) - f(x) - \langle \nabla f(x), z \rangle) \frac{\bar{k}_0}{|z|^{d+\alpha}}\, dz.$$

Additionally, when $b_0(x) \equiv 0$ for all $x \in \mathbb{R}^d$ (in particular, in balanced case: $k(x,z) = k(x,-z)$ for all $x, z \in \mathbb{R}^d$), then $\bar{b}_0 = 0$.

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$$b_0(x) := \frac{1}{2} \int z \frac{(k(x,z) - k(x,-z))}{|z|^{d+\alpha}}\, dz, \quad x \in \mathbb{R}^d.$$

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If $\alpha \in (1,2)$, there exist a vector $\bar{b}_0 \in \mathbb{R}^d$ and a constant $\bar{k}_0 > 0$ such that the process

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converges, as $\varepsilon \to 0$, in the Skorokhod topology to a rotationally invariant α-stable Lévy process $\bar{X}$ with Lévy measure $\frac{\bar{k}_0}{|z|^{d+\alpha}}\,dz$.

- Let $X^{\mathbb{T}^d}$ be the projection of the process X from $\mathbb{R}^d$ to $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$. Then, $X^{\mathbb{T}^d}$ has a unique invariable probability measure $\bar{\mu}(dx)$. Moreover,

$$\bar{b}_0 = \int_{\mathbb{T}^d} b_0(x)\,\bar{\mu}(dx), \quad \bar{k}_0 = \iint_{\mathbb{T}^d \times \mathbb{T}^d} k(y,z)\,dz\,\bar{\mu}(dy).$$

- Central limit theorem for stable laws. Non-central limit theorem when $\alpha \in (1,2)$.

Main result

Theorem

If $\alpha \in (1,2)$, there exist a vector $\bar{b}_0 \in \mathbb{R}^d$ and a constant $\bar{k}_0 > 0$ such that the process

$$\{\varepsilon(X_{\varepsilon^{-\alpha}t} - \varepsilon^{-\alpha}\bar{b}_0 t)\}_{t\geqslant 0}$$

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If $\alpha \in (1,2)$, there exist a vector $\bar{b}_0 \in \mathbb{R}^d$ and a constant $\bar{k}_0 > 0$ such that the process

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- Central limit theorem for stable laws. Non-central limit theorem when $\alpha \in (1,2)$.

Main result

Theorem

If $\alpha = 1$, there exist a vector $\bar{b}_0^\varepsilon \in \mathbb{R}^d$ and a constant $\bar{k}_0 > 0$ such that the process

$$\{\varepsilon X_{\varepsilon^{-1}t} - \bar{b}_0^\varepsilon t)\}_{t \geqslant 0}$$

converges, as $\varepsilon \to 0$, in the Skorokhod topology to a rotationally invariant α-stable Lévy process $\bar{X}$ with Lévy measure $\frac{\bar{k}_0}{|z|^{d+1}}\,dz$.

- $$\bar{b}_0^\varepsilon = \int_{\mathbb{T}^d} b_0^\varepsilon(x)\,\bar{\mu}(dx), \quad \bar{k}_0 = \iint_{\mathbb{T}^d \times \mathbb{T}^d} k(y,z)\,dz\,\bar{\mu}(dy).$$
- If $\bar{b}_0^\varepsilon \to \bar{b}_0$ as $\varepsilon \to 0$, then $\varepsilon X_{\varepsilon^{-1}\cdot} \to \bar{X}_\cdot$ with corresponding infinitesimal generator

$$\bar{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))\,\frac{\bar{k}_0}{|z|^{d+1}}dz + \langle \nabla f(x), \bar{b}_0 \rangle.$$

Thank you for your attention!